S.-T. Yau College Student Mathematics Contests 2026

Analysis and Differential Equations

(5 problems)

Problem 1. For any $n \in \mathbb{N}$, define

$$I_n := \frac{1}{n!} \int_{-\pi/2}^{\pi/2} \left(\frac{\pi^2}{4} - t^2 \right)^n \cos t \, dt.$$

(a) Prove

$$I_{n+1} = 2(2n+1) I_n - \pi^2 I_{n-1}.$$

(b) Show that $\pi^2 \notin \mathbb{Q}$, that is, π^2 is irrational.

Problem 2. For any $n \times n$ complex matrix X, define

$$e^X = \sum_{m \geq 0} \frac{X^m}{m!}.$$

(a) If X, Y are $n \times n$ complex matrices satisfying $[X, Y] = 0$, then

$$e^X e^Y = e^{X+Y} = e^Y e^X.$$

(b) If X, Y are $n \times n$ complex matrices satisfying

$$[X, [X, Y]] = [Y, [X, Y]] = 0,$$

then

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]}.$$

Problem 3. Let $f(z)$ be a smooth function on $|z| < R$, and fix a real number ρ such that $0 < \rho < R$. Put

$$g(z) = \frac{1}{2\pi\sqrt{-1}} \iint_{|\zeta| \leq \rho} \frac{f(\zeta)}{\zeta - z} d\zeta \wedge d\bar{\zeta}.$$

Then

$$\frac{\partial}{\partial \bar{z}} g(z) = f(z), \quad |z| < \rho.$$

Problem 4. Let $C([0,1])$ be the space of continuous functions over $[0,1]$. Consider the ∞-norm on $C([0,1])$

$$||f||_{L^\infty} := \max_{x \in [0,1]} |f(x)|.$$

Then $(C([0,1]), || \cdot ||_{L^\infty})$ is a Banach space. Consider the closed linear subspace $\mathcal{P}$ of $C([0,1])$ consisting of polynomials over $[0,1]$. Show that $\dim \mathcal{P} < \infty$.

Problem 5. Consider the heat equation with periodic boundary conditions on the unit interval $[0,1]$

$$\begin{cases} \partial_t u(t,x) - \partial_{xx} u(t,x) = 0, \ \forall t > 0, \ \forall x \in (0,1); \\ u(t,0) = u(t,1), \ \forall t > 0; \\ \partial_x u(t,0) = \partial_x u(t,1), \ \forall t > 0. \end{cases} \tag{1}$$

(1). Write an orthonormal basis for $L^2([0,1])$, you do not need to verify this. Show that the double derivative operator $f \mapsto f''$ with domain $\mathcal{F} := \{f \in H^2([0,1]) : f(0) = f(1), \ f'(0) = f'(1)\} \subset L^2([0,1])$ is a self-adjoint operator.

(2). Recall that the *fundamental solution* of the heat equation (1) refers to a function $h(t,x,y)$ on $(0,\infty) \times [0,1] \times [0,1]$ which satisfies that

(a) for any $y \in [0,1]$, $(t,x) \mapsto h(t,x,y)$ is a solution of (1);

(b) for any $x \in [0,1]$, $h(t,x,y) \to \delta_x$ weakly as $t \to 0$. That is, for any continuous function φ on $[0,1]$ with $\varphi(0) = \varphi(1)$, $\int_0^1 h(t,x,y)\varphi(y)\,dy \to \varphi(x)$ as $t \to 0$.

Find the fundamental solution $h(t,x,y)$ of the heat equation (1) and check that it satisfies the conditions (a) and (b).

(3). For each fixed $t > 0$, where does the fundamental solution $h(t,x,y)$ attain its maximum (that is, for what $x, y \in [0,1]$ is h maximum)? What can you say about $h(t,x,y)$ when $t \to \infty$? Finally, show that $h(t,x,y) \sim t^{-1/2} \exp\left(-d(x,y)^2/t\right)$ for small $t > 0$, where $d(x,y) := \min\{|x-y|, 1-|x-y|\}$. That is, there exist constants $A, B, C, D > 0$, such that for all $t \in (0,1)$, $x, y \in [0,1]$,

$$\frac{A}{\sqrt{t}} e^{-\frac{Bd(x,y)^2}{t}} \le h(t,x,y) \le \frac{C}{\sqrt{t}} e^{-\frac{Dd(x,y)^2}{t}}.$$

(Hint: There are multiple ways to solve this problem. You might use the fact that the Gaussian function $p(t,x,y) = (4\pi t)^{-1/2} \exp\left(-|x-y|^2/4t\right)$ is the fundamental solution of the heat equation on the real line. The Poisson summation formula can also be useful, depending on the approach.)